Math 522 Exam 3 Solutions

1. For all $k \in \mathbb{N}$, find all real solutions x to the equation $\binom{x}{k} = \binom{x}{k+1}$.

   We first use the definition of the binomial coefficient to get $\frac{1}{k!}x^{\underline{k}} = \frac{1}{k!}x(x-1)\cdots(x-k+1) = \frac{1}{(k+1)!}x(x-1)\cdots(x-k+1)(x-k)$. Multiplying by $(k+1)!$ on both sides, we get $(k+1)x^{\underline{k}} = (x-k)x^{\underline{k}}$. Moving to one side we get $(x-2k-1)x^{\underline{k}} = 0$. This has $k+1$ solutions, namely $x = 0, 1, \ldots, (k-1), (2k+1)$.

2. Use the binomial theorem to prove that the following identity holds for all even $n \in \mathbb{N}_0$:

$$1 - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} - \cdots + \binom{n}{n-2} = n - 1$$

   We use the binomial theorem with $x = -1, y = 1$ to get $0^n = 1 - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} - \cdots + (-1)^{n-2}\binom{n}{n-2} + (-1)^{n-1}\binom{n}{n-1} + (-1)^n\binom{n}{n} = LHS - n + 1$, where we need n to be even so that $(-1)^{n-1} = -1$ and $(-1)^n = 1$.